Scalar field cosmology with non-minimal coupling – perspectives for a new description of evolution of the Universe

Orest Hrycyna and Marek Szydłowski

Department of Theoretical Physics, Faculty of Philosophy, The John Paul II Catholic University of Lublin, Al. Racławickie 14, 20-950 Lublin, Poland
Astronomical Observatory, Jagiellonian University, Orla 171, 30-244 Kraków, Poland
Mark Kac Complex Systems Research Centre, Jagiellonian University, Reymonta 4, 30-059 Kraków, Poland

June 30, 2010

Talk based on the following papers:

- Marek Szydlowski and Orest Hrycyna *Scalar field cosmology in the energy phase-space – unified description of dynamics* JCAP01(2009)039, arXiv:0811.1493 [astro-ph]
- Orest Hrycyna and Marek Szydlowski *Twister quintessence scenario* arXiv:0906.0335 [astro-ph]
- Orest Hrycyna and Marek Szydlowski *Three steps to accelerated expansion* Annalen Phys. 19, 320 (2010), arXiv:0911.2208 [astro-ph.CO]
- Orest Hrycyna and Marek Szydlowski *Unique evolution with non-minimal coupling* (to be published)

Introduction

In the model under consideration we assume the spatially flat FRW universe filled with the non-minimally coupled scalar field and barotropic fluid with the equation of the state coefficient w_m. The action assumes following form

$$S = \frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left(\frac{1}{\kappa^2}R - \varepsilon\Big(g^{\mu\nu}\partial_\mu\phi\partial_\nu\phi + \xi R\phi^2\Big) - 2U(\phi)\right) + S_m, \tag{1}$$

where $\kappa^2 = 8\pi G$, $\varepsilon = +1, -1$ corresponds to canonical and phantom scalar field, respectively, the metric signature is $(-,+,+,+)$, $R = 6\left(\frac{\ddot{a}}{a} + \frac{\dot{a}}{a}\right)$ is the Ricci scalar, a is the scale factor and a dot denotes differentiation with respect to the cosmological time and $U(\phi)$ is the scalar field potential function. S_m is the action for the barotropic matter part.

The dynamical equation for the scalar field we can obtain from the variation $\delta S/\delta\phi = 0$

$$\ddot{\phi} + 3H\dot{\phi} + \xi R\phi + \varepsilon U'(\phi) = 0, \tag{2}$$

and energy conservation condition from the variation $\delta S/\delta g = 0$

$$\mathcal{E} = \varepsilon\frac{1}{2}\dot{\phi}^2 + \varepsilon 3\xi H^2\phi^2 + \varepsilon 3\xi H(\phi^2)^{\cdot} + U(\phi) + \rho_m - \frac{3}{\kappa^2}H^2. \tag{3}$$

Then conservation conditions read

$$\frac{3}{\kappa^2}H^2 = \rho_\phi + \rho_m, \tag{4}$$

$$\dot{H} = -\frac{\kappa^2}{2}\Big[(\rho_\phi + p_\phi) + \rho_m(1 + w_m)\Big] \tag{5}$$

where the energy density and the pressure of the scalar field are

$$\rho_\phi = \varepsilon\frac{1}{2}\dot{\phi}^2 + U(\phi) + \varepsilon 3\xi H^2\phi^2 + \varepsilon 3\xi H(\phi^2)^{\cdot}, \tag{6}$$

$$p_\phi = \varepsilon\frac{1}{2}(1 - 4\xi)\dot{\phi}^2 - U(\phi) + \varepsilon\xi H(\phi^2)^{\cdot} - \varepsilon 2\xi(1 - 6\xi)\dot{H}\phi^2 - \tag{7}$$

$$\varepsilon 3\xi(1 - 8\xi)H^2\phi^2 + 2\xi\phi U'(\phi). \tag{8}$$

Dynamical variables

In what follows we introduce the energy phase space variables

$$x \equiv \frac{\kappa \dot{\phi}}{\sqrt{6}H}, \quad y \equiv \frac{\kappa \sqrt{U(\phi)}}{\sqrt{3}H}, \quad z \equiv \frac{\kappa}{\sqrt{6}}\phi, \tag{9}$$

which are suggested by the conservation condition

$$\frac{\kappa^2}{3H^2}\rho_\phi + \frac{\kappa^2}{3H^2}\rho_m = \Omega_\phi + \Omega_m = 1 \tag{10}$$

or in terms of the newly introduced variables

$$\Omega_\phi = y^2 + \varepsilon\left[(1-6\xi)x^2 + 6\xi(x+z)^2\right] = 1 - \Omega_m. \tag{11}$$

The acceleration equation can be rewritten to the form

$$\dot{H} = -\frac{\kappa^2}{2}\left(\rho_{\mathrm{eff}} + p_{\mathrm{eff}}\right) = -\frac{3}{2}H^2(1 + w_{\mathrm{eff}}) \qquad (12)$$

where the effective equation of the state parameter reads

$$\begin{aligned} w_{\mathrm{eff}} = \frac{1}{1 - \varepsilon 6\xi(1-6\xi)z^2}\Big[& -1 + \varepsilon(1-6\xi)(1-w_m)x^2 + \\ & \varepsilon 2\xi(1-3w_m)(x+z)^2 + (1+w_m)(1-y^2) - \\ & \varepsilon 2\xi(1-6\xi)z^2 - 2\xi\lambda y^2 z\Big] \end{aligned} \qquad (13)$$

where $\lambda = -\frac{\sqrt{6}}{\kappa}\frac{1}{U(\phi)}\frac{\mathrm{d}U(\phi)}{\mathrm{d}\phi}$.

Dynamical system

$$\begin{aligned}
x' &= -(x - \varepsilon\frac{1}{2}\lambda y^2)\Big[1 - \varepsilon 6\xi(1-6\xi)z^2\Big] + \\
&+\frac{3}{2}(x + 6\xi z)\Big[-\frac{4}{3} - 2\xi\lambda y^2 z + \varepsilon(1-6\xi)(1-w_m)x^2 + \\
&+\varepsilon 2\xi(1-3w_m)(x+z)^2 + (1+w_m)(1-y^2)\Big], \qquad (14a) \\
y' &= y\left(2 - \frac{1}{2}\lambda x\right)\Big[1 - \varepsilon 6\xi(1-6\xi)z^2\Big] + \\
&+\frac{3}{2}y\Big[-\frac{4}{3} - 2\xi\lambda y^2 z + \varepsilon(1-6\xi)(1-w_m)x^2 + \\
&+\varepsilon 2\xi(1-3w_m)(x+z)^2 + (1+w_m)(1-y^2)\Big], \qquad (14b) \\
z' &= x\Big[1 - \varepsilon 6\xi(1-6\xi)z^2\Big], \qquad (14c) \\
\lambda' &= -\lambda^2(\Gamma - 1)\,x\Big[1 - \varepsilon 6\xi(1-6\xi)z^2\Big]. \qquad (14d)
\end{aligned}$$

where $\Gamma = \frac{\frac{\mathrm{d}^2 U(\phi)}{\mathrm{d}\phi^2} U(\phi)}{\left(\frac{\mathrm{d}U(\phi)}{\mathrm{d}\phi}\right)^2}$ and prime denotes differentiation with respect to time τ defined as

$$\frac{\mathrm{d}}{\mathrm{d}\tau} = \left[1 - \varepsilon 6\xi(1-6\xi)z^2\right]\frac{\mathrm{d}}{\mathrm{d}\ln a}. \tag{15}$$

To investigate the dynamics of universe described by the dynamical system (14) we need to define a unknown function Γ, i.e. we need to define the potential function $U(\phi)$. In the special cases of the system with the cosmological constant or exponential potential, $U = U_0 = \text{const.}$ or $U = U_0 \exp(-\lambda\phi)$, the dynamical system (14) can be reduced to the 3-dimensional one due to the relation that in the first case we have $\lambda = 0$ and $\Gamma = 0$ and in the second case $\lambda = \text{const.}$ and $\Gamma = 1$. Then dynamical system consists of three equations (14a, 14b, 14c).

There is another possibility of reduction of the system (14) form 4-dimensional dynamical system to 3-dimensional one. If we assume that $z = z(\lambda)$ and $\Gamma = \Gamma(\lambda)$, then using (14c) and (14d) we can find the function $z(\lambda)$ from the differential equation

$$\frac{\mathrm{d}z(\lambda)}{\mathrm{d}\lambda} = z'(\lambda) = -\frac{1}{\lambda^2\big(\Gamma(\lambda) - 1\big)} \tag{16}$$

which can be integrated for some given function $\Gamma(\lambda)$

$$z(\lambda) = -\int \frac{\mathrm{d}\lambda}{\lambda^2\big(\Gamma(\lambda) - 1\big)}. \tag{17}$$

For example if the function $\Gamma(\lambda)$ is assumed in the following form

$$\Gamma(\lambda) = 1 - \frac{1}{\lambda^2}\left(\alpha + \beta\lambda + \gamma\lambda^2\right),$$

then in Table 1 we have gathered forms of the functions $z(\lambda)$ and corresponding potential functions $U(\phi)$ for various configurations of values of parameters α, β and γ. As we see there are various potential functions which are the most common used in the literature of the subject.

Table: Different examples of potential functions for various configurations of parameters values of the assumed form of the $\Gamma(\lambda)$ function $\Gamma(\lambda) = 1 - \frac{1}{\lambda^2}\left(\alpha + \beta\lambda + \gamma\lambda^2\right)$.

parameters	$z(\lambda)$	potential function $U(\phi)$
$\alpha \neq 0,\ \beta = 0,\ \gamma = 0$	$\frac{\lambda}{\alpha} + \mathrm{const.}$	$U_0 \exp\left(-\frac{\alpha}{2}\phi^2 + \mathrm{const.}\phi\right)$
$\alpha = 0,\ \beta \neq 0,\ \gamma = 0$	$\frac{\ln\lambda}{\beta} + \mathrm{const.}$	$U_0 \exp\left(\frac{\mathrm{const.}}{\beta}\exp(\beta\phi)\right)$
$\alpha = 0,\ \beta = 0,\ \gamma \neq 0$	$-\frac{1}{\gamma\lambda} + \mathrm{const.}$	$U_0\left(\gamma\phi - \mathrm{const.}\right)^{\frac{1}{\gamma}}$
$\alpha \neq 0,\ \beta \neq 0,\ \gamma = 0$	$\frac{\ln(\alpha+\beta\lambda)}{\beta} + \mathrm{const.}$	$U_0 \exp\left(\frac{1}{\beta}\left(\alpha\phi + \mathrm{conts.}\exp(\beta\phi)\right)\right)$
$\alpha \neq 0,\ \beta = 0,\ \gamma \neq 0$	$\frac{\arctan\left(\sqrt{\frac{\gamma}{\alpha}}\lambda\right)}{\sqrt{\alpha\gamma}} + \mathrm{const.}$	$U_0\left(\cos\left(\sqrt{\alpha\gamma}(\phi - \mathrm{const.})\right)\right)^{\frac{1}{\gamma}}$
$\alpha = 0,\ \beta \neq 0,\ \gamma \neq 0$	$\frac{\ln\lambda - \ln(\beta+\gamma\lambda)}{\beta} + \mathrm{const.}$	$U_0\left(\exp(\mathrm{const.}\beta) + \gamma\exp(\beta\phi)\right)^{\frac{1}{\gamma}}$
$\alpha \neq 0,\ \beta \neq 0,\ \gamma \neq 0$	$\frac{2\arctan\left(\frac{\beta+2\gamma\lambda}{\sqrt{-\beta^2+4\alpha\gamma}}\right)}{\sqrt{-\beta^2+4\alpha\gamma}} + \mathrm{co.}$	$U_0\exp\left(\frac{\beta}{2\gamma}\phi\right)\left(\cos\left(\frac{1}{2}\sqrt{-\beta^2+4\alpha\gamma}(\phi - \mathrm{co.})\right)\right)^{\frac{1}{\gamma}}$

Of course this simple ansatz for the function $\Gamma(\lambda)$ does not manage all possible potential functions. Let us consider the following function

$$\Gamma(\lambda) = \frac{3}{4} - \frac{\sigma^2\lambda^2}{4\left(2+\sqrt{4\pm\sigma^2\lambda^2}\right)^2}$$

as one can check from (17) we receive

$$z(\lambda) = -\frac{2+\sqrt{4\pm\sigma^2\lambda^2}}{\lambda} + \text{const.}$$

and this example corresponds to the Higgs potential

$$U(\phi) = U_0\Big((\phi - \text{const.})^2 - \sigma^2\Big)^2.$$

We need to stress that the discussion presented below is not restricted to the specific potential function but is generic in the sense that it is valid for any function $\Gamma(\lambda)$ for which integral defined in (17) exists.

Dynamical system II

Then the dynamical system describing the investigated models is in the following form

$$
\begin{aligned}
x' &= -(x - \varepsilon\frac{1}{2}\lambda y^2)\left[1 - \varepsilon 6\xi(1-6\xi)z(\lambda)^2\right] + \\
&\frac{3}{2}(x + 6\xi z(\lambda))\left[-\frac{4}{3} - 2\xi\lambda y^2 z(\lambda) + +\varepsilon(1-6\xi)(1-w_m)x^2 + \right. \\
&\left. +\varepsilon 2\xi(1-3w_m)(x+z(\lambda))^2 + (1+w_m)(1-y^2)\right], \qquad (18a)
\end{aligned}
$$

$$
\begin{aligned}
y' &= y\left(2 - \frac{1}{2}\lambda x\right)\left[1 - \varepsilon 6\xi(1-6\xi)z(\lambda)^2\right] + \\
&+\frac{3}{2}y\left[-\frac{4}{3} - 2\xi\lambda y^2 z(\lambda) + \varepsilon(1-6\xi)(1-w_m)x^2 + \right. \\
&\left. +\varepsilon 2\xi(1-3w_m)(x+z(\lambda))^2 + (1+w_m)(1-y^2)\right], \qquad (18b)
\end{aligned}
$$

$$
\lambda' = -\lambda^2(\Gamma(\lambda)-1)\,x\left[1 - \varepsilon 6\xi(1-6\xi)z(\lambda)^2\right]. \qquad (18c)
$$

Table: Critical points of the system under consideration.

x^*	y^*	λ^*	w_{eff}
$x_1^* = -6\xi z(\lambda_1^*)$	$y_1^* = 0$	$\lambda_1^*: z(\lambda)^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$	$\pm\infty$
$x_{2a}^* = -6\xi z(\lambda_{2a}^*)$	$(y_{2a}^*)^2 = \frac{4\xi}{2\xi\lambda_{2a}^* z(\lambda_{2a}^*)+(1+w_m)}$	$\lambda_{2a}^*: z(\lambda)^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$	$w_m - 4\xi$
$x_{2b}^* = 0$	$(y_{2b}^*)^2 = \frac{2\xi(1-3w_m)}{(1-6\xi)\left(2\xi\lambda_{2b}^* z(\lambda_{2b}^*)+(1+w_m)\right)}$	$\lambda_{2b}^*: z(\lambda)^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$	$\frac{w_m-2\xi}{1-6\xi}$
$x_{3a}^* : g(x) = 0$ [1]	$y_{3a}^* = 0$	$\lambda_{3a}^*: z(\lambda)^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$	$\frac{1}{3}$
$x_{3b}^* = 0$	$y_{3b}^* = 0$	$\lambda_{3b}^*: z(\lambda)^2 = \frac{1}{\varepsilon 6\xi}$	$\frac{1}{3}$
$x_4^* = 0$	$y_4^* = 0$	$\lambda_4^*: z(\lambda) = 0$	w_m
$x_5^* = 0$	$(y_5^*)^2 = 1 - \varepsilon 6\xi z(\lambda_5^*)^2$	$\lambda_5^*: \lambda z(\lambda)^2 + 4z(\lambda) - \frac{\lambda}{\varepsilon 6\xi} = 0$	-1

[1] $g(x) = \varepsilon(1-4\xi-w_m)x^2 + \varepsilon 4\xi(1-3w_m)z(\lambda_{3a}^*)x + \frac{2\xi}{1-6\xi}(1-3w_m)$

1 Finite scale factor singularity
eigenvalues: $l_1 = 6\xi, \quad l_2 = 6\xi, \quad l_3 = 12\xi,$

- for $\xi > 0$ – unstable node
- for $\xi < 0$ – stable node

2a Fast-roll inflation
eigenvalues : $l_1 = 0, \quad l_2 = 12\xi, \quad l_3 = -12\xi,$

- non-hyperbolic critical point $\rightarrow$ the center manifold theorem

2b Slow-roll inflation
eigenvalues : $l_1 = l_2 = l_3 = 0$

- degenerated critical point

Character of critical points *cont.*

3 Radiation domination epoch generated by non-minimal coupling
two critical points :

- for phantom scalar field and $\xi > 0$ eigenvalues :
  $l_1 = 0, \quad l_2 > 0, \quad l_3 < 0$ – non-hyperbolic critical point
- for canonical scalar field and $\xi > 0$ eigenvalues :
  $l_1 = 6\xi(1-3w_m), \quad l_2 = 12\xi, \quad l_3 = -6\xi$
  $\longrightarrow$ Hartman-Grobman theorem
  $\longrightarrow$ linearised solution
  $\longrightarrow$ $w_{\text{eff}}(z)$ parameterisation

4 Matter domination epoch
eigenvalues :
$$l_{1,2} = -\frac{3}{4}\left((1-w_m) \pm \sqrt{(1-w_m)^2 - \frac{16}{3}\xi(1-3w_m)}\right),$$
$$l_3 = \frac{3}{2}(1+w_m),$$
non-degenerated for $w_m \neq -1$ and $w_m \neq \frac{1}{3}$
$\longrightarrow$ Hartman-Grobman theorem $\longrightarrow$ linearised solution

5 The present accelerated expansion epoch
In the most general case without assuming any specific form of the potential function we are unable to find coordinates of this point. In spite of this we are able to formulate general conditions for stability of this critical point. This requires that the real parts of the eigenvalues of the linearization matrix calculated at this point must be negative. From the Routh-Hurwitz test we have that the following conditions should be fulfilled to assure stability of this critical point

$$Re[l_{1,2,3}] < 0 \iff 3\xi \frac{h'(\lambda_5^*)}{z'(\lambda_5^*)}(y_5^*)^2 > 0 \tag{19}$$

where

$$h(\lambda) = \lambda z(\lambda)^2 + 4z(\lambda) - \frac{\lambda}{\varepsilon 6\xi}$$

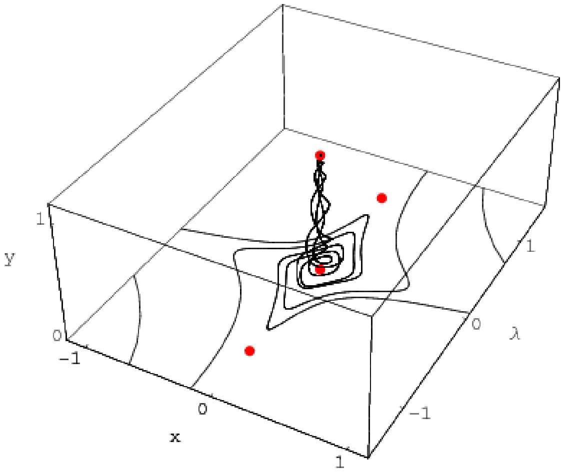

Figure: Three-dimensional phase portrait of the dynamical system under consideration for $\Gamma(\lambda) = \frac{\lambda}{\alpha}$. Trajectories represent a twister type solution which interpolates between the radiation dominated universe (a saddle type critical point), the matter dominated universe (an unstable focus critical point) and the accelerating universe (a stable focus critical point).

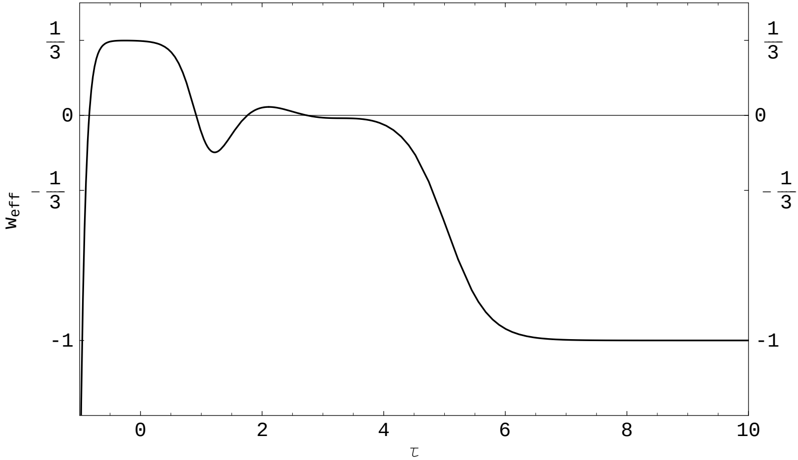

Figure: The evolution of $w_{\rm eff}$ given by the relation (13) for the non-minimally coupled canonical scalar field $\varepsilon = +1$ and the positive coupling constant ξ. The sample trajectory used to plot this relation starts its evolution at $\tau_0 = 0$ near the saddle type critical point ($w_{\rm eff} = 1/3$) and then approaches an unstable focus critical point $w_{\rm eff} = w_m = 0$ and next escapes to the stable deSitter state with $w_{\rm eff} = -1$. The existence of a short time interval during which $w_{\rm eff} \simeq \frac{1}{3}$ is the effect of the nonzero coupling constant ξ.

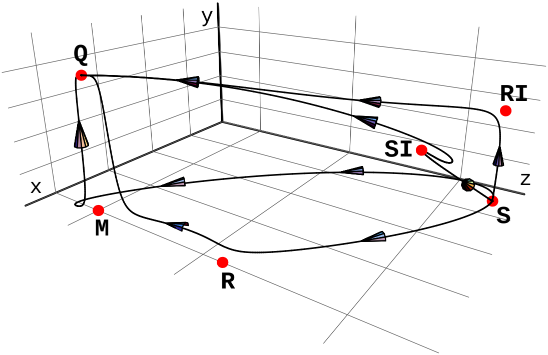

Figure: The phase space portrait for the model with cosmological constant and the canonical scalar field ($\varepsilon = +1$) with $\xi = 1/8$ and the dust matter $w_m = 0$. The critical points are: S – the finite scale factor singularity, RI – the rapid-roll inflation, SI – the slow-roll inflation, R – the radiation dominated era, M – the barotropic matter dominated era and Q – the quintessence era. Note that the critical points representing the finite scale factor singularity, the rapid-roll inflation and the slow-roll inflation have the same value of z coordinate.

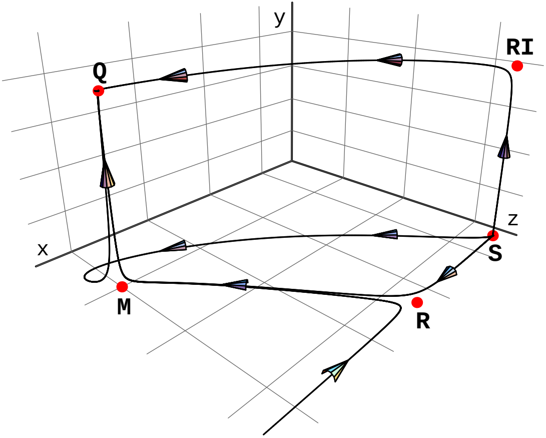

Figure: The phase space portrait for the model with cosmological constant and the phantom scalar field ($\varepsilon = -1$) with $\xi = 1/4$ and the dust matter $w_m = 0$. The critical points are: S – the finite scale factor singularity, RI – the rapid-roll inflation, R – the radiation dominated era, M – the barotropic matter dominated era and Q – the quintessence era. In the case of the phantom scalar field the critical point representing slow-roll inflation is not present. The critical pints denoted as S, RI and R have the same value of z coordinate.

Conclusions

- We pointed out the presence of the new interesting solution for the non-minimally coupled scalar field cosmology which we called the twister solution (because of the shape of the corresponding trajectory in the phase space).
- This type of the solution is very interesting because in the phase space it represents the 3-dimensional trajectory which interpolates different stages of evolution of the universe, namely, the radiation dominated, dust filled and accelerating universe.
- We are able to find linearised solutions around all these intermediate phases, and hence, parameterisations for $w_{\text{eff}}(a)$ in different epochs of the universe history.
- It is interesting that the presented structure of the phase space is allowed only for non-zero value of coupling constant, therefore it is a specific feature of the non-minimally coupled scalar field cosmology.