Investment delay and economic system dynamics

Adam Krawiec [1,3] and Marek Szydłowski [2,3]

[1] Institute of Economics & Management, Jagiellonian University
[2] Institute of Philosophy of Nature and Natural Sciences, John Paul II Catholic University
[3] Mark Kac Complex Research Centre, Jagiellonian University
uoszydlo@cyf-kr.edu.pl, uukrawie@cyf-kr.edu.pl

Abstract

We study the role of investment on dynamics of macroeconomic models. The delay of investment is understand as the time to build of capital goods. The result is delay differential equations applied to modelling the economic growth [3, 6]. The time lag in the investment is responsible for complexity of economic evolution in short and long run. The long run effects determine characteristics of economic development. The possibility of description of irregularities forced by delay in growth paths is considered. We also investigate optimal consumption in terms of advanced-retarded differential equations.

1. Introduction

We present two models of economic phenomena. First, the model of business cycle where a time delay in investment process causes endogenous cycles to reproduce economic fluctuations. Second, the model of economic growth where production lags are responsible for the irregularities on the long-run growth path. The idea of using delay differential equations to model business cycles was proposed by Kalecki [2]. He considered a gestation period between taking investment decisions and delivering capital goods, and showed formally that endogenous cycles can be generated. To study of nonlinear behaviour of models we use techniques of functional analysis and bifurcation theory. For determining the cyclic behaviour the Poincaré-Andronov-Hopf bifurcation theorem is used which has been generalized for delay differential equations [1]. The Hopf theorem predicts bifurcation to a limit cycle when (a) the real part of a pair of complex conjugate (with nonzero imaginary part) eigenvalues λ of the characteristic equation changes its sign from negative to the positive as a bifurcation parameter T is varied, and (b) the derivative of the real part of the eigenvalue with the respect to the parameter T is positive as real part σ passes through zero

$$\frac{d}{dT}\operatorname{Re}\lambda(T)|_{T=T_{\mathrm{bi}}} > 0. \tag{1}$$

2. The model of business cycle

We formulate the Kaldor-Kalecki model of business cycle as the time-delay differential equation system [3, 7, 5, 8]

$$\frac{dY}{dt} = \alpha[I(Y(t),K(t)) - S(Y(t),K(t))] \tag{2}$$

$$\frac{dK}{dt} = I(Y(t-T),K(t)) - \delta K(t). \tag{3}$$

where I is the investment and S is the saving function, Y is gross product, K is capital stock, α is the adjustment coefficient in the goods market, δ is the depreciation rate of the capital stock, and the time delay T is a constant average lag between investment decision and implementation of new capital.

The saving function S depends only on Y and is linear such that $S_Y = \gamma \in (0,1)$. Additionally, we assume that the investment function $I(Y,K)$ separates in respect to its two arguments and both are linear such that $I_Y > 0$, $I_K = \beta < 0$ then $I(Y,K) = \eta Y + \beta K$. With these assumptions the Kaldor-Kalecki dynamical system has the form

$$\dot{Y} \equiv \frac{dY}{dt} = \alpha\eta Y(t) + \alpha\beta K(t) - \alpha\gamma Y(t) \tag{4}$$

$$\dot{K} \equiv \frac{dK}{dt} = \eta Y(t-T) + (\beta-\delta)K(t) \tag{5}$$

or equivalently

$$\ddot{Y}(t) + f(Y)\dot{Y}(t) + g(Y) = 0, \tag{6}$$

where

$$f(Y(t)) = -\alpha\frac{\partial I}{\partial Y} + \alpha\gamma - \beta + \delta$$

$$g(Y(t),Y(t-T)) = \alpha(\eta-\gamma)(\beta-\delta)Y(t) - \alpha\beta\eta Y(t-T).$$

If we assume a solution of eq. (6) in the form $Y(t) = e^{\lambda t}$, we derive the eigenvalue equation

$$\lambda^2 + A\lambda + B + De^{-\lambda T} = 0 \tag{7}$$

where A, B and D are constant

$$A = -\alpha\eta + \alpha\gamma - (\beta-\delta)$$

$$B = \alpha(\eta-\gamma)(\beta-\delta)$$

$$D = -\alpha\beta\eta.$$

We assume the eigenvalue in the form $\lambda = \sigma + i\omega$ and write the real and imaginary parts of the eigenvalue equation as

$$\sigma^2 - \omega^2 + \sigma A + B + De^{-\sigma T}\cos\omega T = 0 \tag{8}$$

$$2\sigma\omega + \omega A - De^{-\sigma T}\sin\omega T = 0 \tag{9}$$

The limit cycle bifurcation occurs when the real part of a complex conjugate pair of eigenvalues changes its sign from negative to positive. Putting $\sigma = 0$ and $y = \omega T$, first we solve eq. (9) then eq. (8) and obtain

$$T_{\mathrm{bif}} = \frac{A}{D}\frac{y}{\sin y} = \frac{A}{D}\frac{\arccos z}{\sin(\arccos z)} = \frac{A}{D}\frac{\arccos z}{\frac{\pi}{2} - z}$$

where

$$z_{1,2} = \frac{1}{2D}\left(-A^2 \mp \sqrt{A^4 - 4(A^2B - D^2)}\right).$$

We saw that condition (a) occurs in the Kaldor-Kalecki model as the time-delay parameter is increased. We can verify condition (b) by differentiating the eigenvalue equation (7), which yields

$$\frac{\partial\lambda}{\partial T} = \frac{\lambda De^{-\lambda T}}{2\lambda + A - DTe^{-\lambda T}}$$

and it follows

$$\left.\frac{\partial\sigma}{\partial T}\right|_{T=T_{\mathrm{bif}}} = \frac{D(A\omega\sin\omega T + 2\omega^2\cos\omega T)}{(2\omega + TD\sin\omega T)^2 + (A - TD\cos\omega T)^2}$$

If we assume that $A/D \geq 0$ (D is always positive) then $\frac{\partial\sigma}{\partial T}|_{T=T_{\mathrm{bif}}}$ is positive for the argument $y = \omega T \in [-\frac{\pi}{2}, \frac{\pi}{2}]$. Moreover we can determine the period of an orbit by

$$P = \frac{2\pi}{|\lambda(T_{\mathrm{bif}})|} = \frac{2\pi T_{\mathrm{bif}}}{|y|}.$$

In Fig. 1 it is shown the dependence of the bifurcation parameter on z. And the amplitude of this relation is equal A/D and corresponds the value of $z = 0$. In the right panel it is checked that the derivative of a real part of the eigenvalue with respect to T is positive for small y.

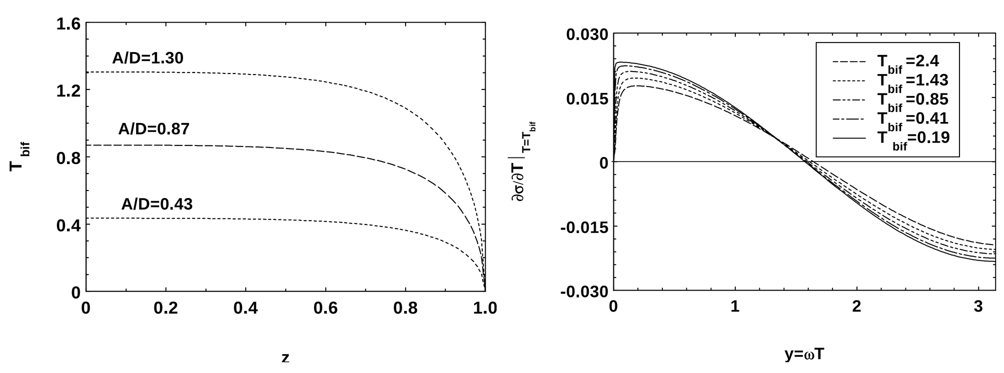

Figure 1: *The dependence of bifurcation parameter T_{bif} on $z = \cos y$ (left). The dependence of $\frac{\partial\sigma}{\partial T}|_{T_{\mathrm{bif}}}$ on y (right).*

3. The model of economic growth

Assuming that the rate of change of the capital stock at moment T is a function of the productive capital stock at $t - T$, and capital stock depreciates at the rate $\delta \in [0,1]$ we obtain the following dynamic equation [9, 6]

$$\dot{k} = sf(k(t-T)) - \delta k(t-T), \tag{10}$$

$$k(t) = \phi(t) \quad \text{for} \quad t \in [-T, 0],$$

where $f(k)$ is a neoclassical production function (that is continuous, increasing, and strictly concave in variable k), and $s \in (0,1)$ is a constant saving rate. Instead of an initial point value for an ordinary differential equation, the initial function $\phi(t)$ is required which is defined over the range of time delimited by the delay.

Let start from the local stability analysis and consider the unique critical point k^* for which

$$sf(k^*) = \delta k^*. \tag{11}$$

Such a point exists if $f(k)$ satisfies the Inada conditions. After the linearization of the right-hand side of system (10) at the critical point we obtain

$$\dot{k}(t) = sf'(k)|_{k=k^*}(k - k^*)(t-T) - \delta(k - k^*)(t-T). \tag{12}$$

We shift the origin to the critical point by introducing the new variable $z(t) = k - k^*$, then eq. (12) can be rewritten as

$$\dot{z} = [sf'(k)|_{k=k^*} - \delta]z(t-T). \tag{13}$$

To find the presence of cyclic behaviour we assume $k(t) = e^{\lambda t}$ and consider the characteristic equation for eq. (13)

$$h(\lambda) = \lambda - Ae^{-\lambda T} = 0. \tag{14}$$

To check whether a limit cycle is created by the Hopf bifurcation, we, first, find the bifurcation value of the time delay parameter, $T = T_{\mathrm{bi}}$. For this purpose it is sufficient to show the existence of a unique pair of the complex conjugated solutions of (14) $(\lambda, \bar{\lambda})$. After applying Euler's formula, and decomposing the characteristic equation (14) into real and imaginary parts of an eigenvalue $\lambda = \mu + i\omega$ we obtain

$$\mu - Ae^{-\mu T}\cos\omega T = 0 \tag{15a}$$

$$\omega + Ae^{-\mu T}\sin\omega T = 0. \tag{15b}$$

To find the Hopf cycle we look for $(\omega_{\mathrm{bi}}, T_{\mathrm{bi}})$ when a pair of roots of (15) is purely imaginary, i.e. $\mu = 0$. Given $A < 0$ by assumption we obtain the following solution

$$\omega_{\mathrm{bi}} = -A \quad \text{and} \quad T_{\mathrm{bi}} = -\frac{1}{A}\frac{(4m+1)\pi}{2}, \qquad m = 0,1,2,\ldots. \tag{16}$$

The first order bifurcation ($m = 0$) appears when $T_{\mathrm{bi}} = -\pi/(2A)$ and therefore, we can determine the period of oscillation $P \cong 2\pi/\omega$ as

$$P = -\frac{2\pi}{A} = 4T_{\mathrm{bi}} > 0. \tag{17}$$

Note that m appears only in the calculation of T_{bi}. Therefore, there is a unique value of ω, and consequently, the period P corresponds to a periodic orbit. We obtain one distinguished period, called the main cycle, which has the economic meaning because its period is longer than the bifurcation value of time delay parameter and all other cycles with $m \geq 1$ are excluded. The period of the main cycle depends only on A, i.e., the production function, the saving rate, and the depreciation rate.

The next step in proving the existence of the Hopf bifurcation is checking that there is no other eigenvalues with $\operatorname{Re}\lambda = 0$. If the real value μ exists then from (15) we have

$$\mu = Ae^{-\mu T}, \quad \lambda = \mu. \tag{18}$$

By consideration of the characteristic polynomial $h(\lambda) = \lambda - Ae^{-\lambda T}$ we observe that $h(\lambda) > 0$.

The last step is to check the transversality condition. Differentiation of eq. (14) gives

$$\frac{\partial\lambda}{\partial T} = \operatorname{sign}\left[\frac{\omega^2}{(1+T\mu)^2 + T^2\mu^2}\right] > 0.$$

The confirmation of this property ends our proof of the existence of the Hopf cycle. The limit cycle in the delay Solow model is presented in Fig. 2.

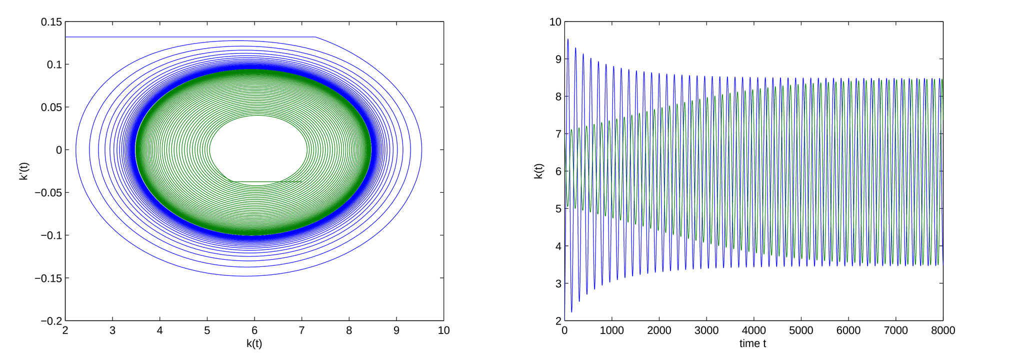

Figure 2: *The limit cycle in the Solow model with time delay.*

4. Conclusion

We studied two economic models described by delay differential equations. The time delay was the period of between the starting and finishing the investment. Both in the model of business cycle and the model of economic growth the cyclic behaviour can appear for some specific values of parameters due to the Hopf bifurcation where the time delay parameter is the bifurcation parameter.

References

[1] J. K. Hale, S. M. Verduyn Lunel. *Introduction to Functional Differential Equations*. Springer, New York 1993.

[2] M. Kalecki. A macrodynamic theory of business cycles. Econometrica, 3:327–344, 1935.

[3] A. Krawiec, M. Szydłowski. The Kaldor-Kalecki business cycle model *Annals of Operations Research*, 89:89-100, 1999.

[4] R. M. Solow. A contribution to the theory of economic growth, *Quarterly Journal of Economics*, 70:65–94, 1956.

[5] M. Szydłowski, A. Krawiec, J. Toboła. Nonlinear oscillations in business cycle model with time lags. *Chaos, Solitons & Fractals*, 12:505-517, 2001.

[6] M. Szydłowski, A. Krawiec. A Note on Kaleckian Lags in the Solow Model. *Review of Political Economy*, 16:501-506, 2004.

[7] M. Szydłowski, A. Krawiec. The Kaldor-Kalecki model of business cycle as a two-dimensional dynamical system. *Journal of Nonlinear Mathematical Physics*, 8, 266-271 Suppl., 2001.

[8] M. Szydłowski, A. Krawiec. The stability problem in the Kaldor-Kalecki business cycle model. *Chaos, Solitons & Fractals*, 25:299-305, 2005.

[9] Zak, P. J. (1999) Kaleckian lags in general equilibrium, *Review of Political Economy*, 11, pp. 321–330.