Investment delay and economic system dynamics

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Abstract

We study the role of investment on dynamics of macroeconomic models. The delay of investment is understood as the time to quill of capital goods. The result is delay differential equations applied to modelling the economic growth [3, 6]. The time lag in the investment is responsible for complexity of economic evolution in short and long run. The long run effects determine characteristics of economic development. The possibility of description of irregularities forced by delay in growth paths is considered. We also investigate optimal consumption in terms of advanced-retardated differential equations.

1. Introduction

We present two models of economic phenomena. First, the model of business cycle where a time delay in investment process causes endogenous cycles to reproduce economic fluctuations. Second, the model of economic growth where production lags are responsible for the irregularities on the long-run growth path. The idea of using delay differential equations to model business cycles was proposed by Kalecki [2]. He considered a gestation period between taking investment decisions and delivering capital goods, and showed formally that endogenous cycles can be generated. To study of nonlinear behaviour of models we use techniques of functional analysis and bifurcation theory. For determining the cyclic behaviour the Poincaré-Andronov-Hopf bifurcation theorem is used which has been generalized for delay differential equations [1]. The Hopf theorem predicts bifurcation to a limit cycle when (a) the real part of a pair of complex conjugate (with nonzero imaginary part) eigenvalues of the characteristic equation changes its sign from negative to positive as a bifurcation parameter \( \tau \) is varied, and (b) the derivative of the real part of the eigenvalue with the respect to the parameter \( \tau \) is positive as real part \( \rho \) passes through zero.

\[ \frac{d\rho}{d\tau} \bigg|_{\tau = \tau_*} = 0. \] (1)

2. The model of business cycle

We formulate the Kaldor-Kalecki model of business cycle as the time-delay differential equation system [3, 7, 8]

\[ \frac{dY}{dt} = \alpha(Y)(Y, K(t)) - S(Y)(Y, K(t)) \] (2)
\[ \frac{dK}{dt} = g(Y(t-T), K(t)) - \delta K(t), \] (3)

where \( Y \) is the investment and \( K \) is the saving function, \( Y \) is gross product, \( K \) is capital stock, \( \alpha \) is the adjustment coefficient in the goods market, \( \delta \) is the depreciation rate of the capital stock, and the time delay \( T \) is a constant average lag between investment decision and implementation of new capital. The saving function \( S \) depends only on \( Y \) and is linear such that \( SY = \gamma \theta (0, 1) \). Additionally, we assume that the investment function \( \alpha(Y, K) \) is separable with respect to its two arguments and both are linear such that \( \alpha \beta \gamma \chi = \theta \) and \( S(Y, K(t)) = \alpha \gamma \delta (Y - \alpha Y). \]

If we assume \( \gamma = 0 \) \( \theta \) in (2) the resulting bifurcation value of time delay parameter and all other cycles distinguished period, called the main cycle, which has the bi- cycle function \( \omega (t) \) as the time delay parameter is increased. We can determine the period of oscillations \( P \approx 2\pi / \omega \) as

\[ P = \frac{2\pi}{\omega} \left[ \frac{1}{4} + \frac{1}{4} \right] > 0. \] (17)

Note that \( m \) appears only in the calculation of \( T_{m0} \). Therefore, there is a unique value of \( \tau \), and consequently, the period \( P \) provides only a periodic orbit. We obtain one distinguished period, called the main cycle, which has the bifunction value of time delay parameter and all other cycles with \( m \) are periodic. The period of the main cycle depends only on \( \alpha \), i.e., the production function, the saving rate, and the depreciation rate.

3. The model of economic growth

Assuming that the rate of change of the capital stock at moment \( t \) is a function of the productive capital stock at \( t-T \) and capital stock depreciates at the rate \( \delta \in [0, 1] \) we obtain the following dynamic system [9, 6]

\[ k(t) = \alpha k(t-T) - \delta k(t) - \beta k(t)^2 \] (10)

where \( \alpha \) is the neoclassical production function (that is continuous, increasing, and strictly concave in variable \( \lambda \), and \( \beta < 0 \)) is a constant saving rate. Instead of an initial point value for an ordinary differential equation, the initial condition \( (k_0, \lambda, \beta) \) is required which is defined over the range of time delimitated by the delay. Let start from local stability analysis and consider the unique critical point \( \lambda^* \) for which

\[ \lambda^* = \beta k^2. \] (11)

Such a point exists if \( g(k_0, \beta) > 0 \) satisfies the Inada conditions. After the linearization of the right-hand side of system (10) at the critical point we obtain

\[ \lambda^* = \alpha k(t)^2 - \delta k(t) - \beta k(t)^2 \] (12)

We start from the point of criticality by introducing the new variable \( x(t) = k(t) - \lambda^* \), then \( \alpha k(t)^2 \) can be rewritten as

\[ - k(t)^2 \] (13)

To check whether a limit cycle is created by the Hopf bifurcation, we first, find the bifunction value of the time delay parameter. \( T_{m0} \). For this purpose it is sufficient to show the existence of a unique pair of the complex conjugated solutions of (14) \( \lambda^* \). Applying Euler's formula, and decomposing the characteristic equation (14) into real and imaginary parts of an eigenvalue \( \lambda^* = \rho + i\omega \) we obtain

\[ \mu = \rho^2 - \omega^2 = 4\rho > 0 \] (15a)

\[ \omega = 2\omega^2 \] (15b)

4. Conclusion

We studied two economic models described by delay differential equations. The time delay was the period of between the starting and finishing the investment. Both in the model of business cycle and the model of economic growth the cyclical behaviour can appear for some specific values of parameters due to the Hopf bifurcation where the time delay parameter is the bifurcation parameter.

References