Dynamics of extended quintessence on the phase plane

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In this publication we investigate dynamics of a flat FRW cosmological model with a non-minimally coupled scalar field with the coupling term $\xi R\psi^2$ in the scalar field action. The quadratic potential function $V(\psi) \propto \psi^2$ is assumed. All the evolutional paths are visualized and classified in the phase plane, at which the parameter of non-minimal coupling ξ plays the role of a control parameter. The fragility of global dynamics with respect to changes of the coupling constant is studied in details. We find that the future big rip singularity appearing in the phantom scalar field cosmological models can be avoided due to non-minimal coupling constant effects. We have shown the existence of a finite scale factor singular point (future or past) where the Hubble function as well as its first cosmological time derivative diverges.

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I. INTRODUCTION

Recently scalar fields have played a very important role in cosmology. They are used in many phenomenological models like quintessence [1, 2]. Scalar fields are also very important in description of dynamics in the loop quantum cosmology, which base on the background independent theory without the canonical notion of time. In this theory one scalar field is chosen as an internal clock for other fields [3]. The scalar fields with a potential function are also very important in modelling of inflation. For example a scalar field with the simplest quadratic potential function was assumed in Linde's conception of chaotic inflation [4]. The 5 years of WMAP observations [5] rejected many

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inflationary scenarios (potential functions $V(\psi)$), while models with a simple quadratic potential are admitted on the 1σ confidence level.

In the standard quintessence energy density of the minimally coupled to gravity scalar field mimics the effective cosmological constant. Of course the detailed evolution is dependent upon a specific form of the potential $V(\psi)$ but the ψ^2 contribution can be treated as a leading order term of expansion of the potential function.

We extend the quintessence scenario by incorporating the non-minimal coupling constant [6, 7, 8, 9, 10]. In this paper we present a phase space analysis of the evolution of a spatially flat Friedmann-Robertson-Walker (FRW) universe containing a non-minimally coupled to gravity scalar field, both canonical and phantom, with the simplest form of quadratic potential function. The similar analysis for the case of minimally coupled scalar field was performed in [11]. We extend this analysis on models with a non-zero coupling constant ξ which plays the role of a control parameter for an autonomous dynamical system on the phase plane. Therefore the location of fixed points (physically representing asymptotic states of the system) as well as their character depends upon the value of ξ. The values of parameter ξ for which the global dynamics changes dramatically are called bifurcation values. In our previous paper [12], in case of conformal coupling and quadratic potential function we have shown that phantom cosmology can be treated as a simple model with a scattering of trajectories whose character depends crucially on the sign of the potential function. We also demonstrated that there is a possibility of chaotic behavior in the flat Universe with a conformally coupled phantom field in the system considered on the non-zero energy level.

The minimally coupled scalar field endowed with a quadratic potential function has a strong motivation in inflationary models and its generalizations with a simple non-minimal coupling term $\xi R\psi^2$ have been studied [13] in the context of the origin of the canonical inflaton field itself. The physical motivation to investigate a non-minimally coupled scalar field cosmological models could be possible application of this models to inflationary cosmology or to the present dark energy (see for example [14, 15, 16, 17, 18, 19, 20, 21, 22, 23, 24, 25, 26, 27, 28, 29]).

The coupling constant ξ is a free parameter of the theory which should be estimated from the observational data. Recently we have shown that distant supernovae can be used to estimate the value of this parameter [10].

The main advantage of dynamical system analysis is that we can visualize all the trajectories of the system admissible for all initial conditions. Therefore one can classify generic routes to the accelerating phase (the de Sitter attractor where $p_\psi = -\rho_\psi$). This attractor corresponds to the model with the cosmological constant.

The paper has a following organization: in section II we reduce dynamics to the form of an autonomous dynamical system which describes both canonical and phantom scalar field models. Section III is devoted to a detailed analysis of the phase portraits for different values of the parameter ξ. In this section we also discuss the change of evolutionary scenarios upon the value of parameter ξ.

II. NON-MINIMALLY COUPLED SCALAR FIELD COSMOLOGIES AS A DYNAMICAL SYSTEM

We assume the flat model with the FRW geometry, i.e. the line element has the form

$$\mathrm{d}s^2 = -\mathrm{d}t^2 + a^2(t)[\mathrm{d}r^2 + r^2(\mathrm{d}\theta^2 + \sin^2\theta \mathrm{d}\varphi^2)], \tag{1}$$

where $0 \leq \varphi \leq 2\pi$, $0 \leq \theta \leq \pi$ and $0 \leq r \leq \infty$ are the comoving coordinates and t stands for the cosmological time.

It is also assumed that a source of gravity is a scalar field ψ with a generic coupling to gravity. The gravitational dynamics is described by the standard Einstein-Hilbert action

$$S_g = \frac{1}{2}m_p^2 \int \mathrm{d}^4x\sqrt{-g}R, \tag{2}$$

the action for the matter source is

$$S_\psi = -\frac{1}{2}\int \mathrm{d}^4x\sqrt{-g}\left[\varepsilon\left(g^{\mu\nu}\psi_\mu\psi_\nu + \xi R\psi^2\right) + 2V(\psi)\right]. \tag{3}$$

where $m_{Pl}^2 = 1/(8\pi G) = 1/\kappa$ and $R = 6(\ddot{a}/a + \dot{a}^2/a^2)$ and $\varepsilon = +1, -1$ corresponds to the scalar field and the phantom scalar field, respectively. For simplicity and without lost of generality we will assume $4\pi G/3 = 1$ which corresponds to $\kappa = 6$.

After dropping the full derivatives with respect to time we obtain the dynamical equation for scalar field from variation $\delta(S_g + S_\psi)/\delta\psi = 0$

$$\ddot{\psi} + 3H\dot{\psi} + \xi R\psi + \varepsilon V'(\psi) = 0. \tag{4}$$

as well as the energy conservation condition from variation $\delta(S_g + S_\psi)/\delta g = 0$

$$\mathcal{E} = \varepsilon\frac{1}{2}\dot{\psi}^2 + \varepsilon 3\xi H^2\psi^2 + \varepsilon 3\xi H(\psi^2)^{\dot{}} + V(\psi) - \frac{3}{\kappa}H^2 \tag{5}$$

If we include other forms of matter this condition can be expressed as

$$\frac{3}{\kappa}H^2 = \rho_\psi + \rho_r + \rho_m \tag{6}$$

where ρ_r and ρ_m are energy densities of radiation and matter, respectively. It can be shown that for any value of ξ scalar field behaves like a perfect fluid with energy density ρ_ψ and pressure p_ψ

$$\rho_\psi = \varepsilon\frac{1}{2}\dot{\psi}^2 + V(\psi) + \varepsilon\xi\big[3H(\psi^2)\dot{} + 3H^2\psi^2\big], \quad (7a)$$

$$p_\psi = \varepsilon\frac{1}{2}\dot{\psi}^2 - V(\psi) - \varepsilon\xi\big[2H(\psi^2)\dot{} + (\psi^2)\ddot{} + (2\dot{H} + 3H^2)\psi^2\big]. \quad (7b)$$

Changing the dynamical variables according to the relation

$$\dot{\psi} = \frac{\mathrm{d}\psi}{\mathrm{d}t} = \frac{\dot{a}}{a}\frac{\mathrm{d}\psi}{\mathrm{d}\ln a} = H\psi'$$

we can express the Hubble function as

$$H^2 = 2\frac{V(\psi) + \rho_r + \rho_m}{\frac{6}{\kappa} - \varepsilon\big[(1-6\xi)\psi'^2 + 6\xi(\psi'+\psi)^2\big]}. \quad (8)$$

The denominator of (8) equal zero denotes a line in the phase space of singularities of the Hubble function which separates the phase space in two regions one physical $H^2 > 0$, and the second one nonphysical $H^2 < 0$. It does not depend on the form of the potential function but only on a value of the coupling constant.

The Euler-Lagrange equations for the system under consideration are given in the form

$$a^2\frac{\mathrm{d}^2\psi}{\mathrm{d}\eta^2} + 6\xi a\psi\frac{\mathrm{d}^2 a}{\mathrm{d}\eta^2} = -2a\frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\psi}{\mathrm{d}\eta} - \varepsilon a^4 V'(\psi), \quad (9a)$$

$$\frac{6}{\kappa}\frac{\mathrm{d}^2 a}{\mathrm{d}\eta^2}(1-\varepsilon\kappa\xi\psi^2) - \varepsilon 6\xi a\psi\frac{\mathrm{d}^2\psi}{\mathrm{d}\eta^2} = -\varepsilon a(1-6\xi)\Big(\frac{\mathrm{d}\psi}{\mathrm{d}\eta}\Big)^2 + \varepsilon 12\xi\psi\frac{\mathrm{d}a}{\mathrm{d}\eta}\frac{\mathrm{d}\psi}{\mathrm{d}\eta} + 4a^3V(\psi) + \rho_{m,0}. \quad (9b)$$

where η stands for the conformal time, $\mathrm{d}\eta = \mathrm{d}t/a$.

After the elimination of the scale factor and its derivative from system (9) we obtain the condition

$$\begin{aligned} &\big(\psi''+\psi'\big)\Big(\frac{6}{\kappa} - \varepsilon 6\xi(1-6\xi)\psi^2\Big) - \varepsilon\psi'^2(1-6\xi)\big(\psi'+6\xi\psi\big) + \\ &+\frac{1}{H^2}\Big\{\varepsilon\frac{6}{\kappa}V'(\psi)\big(1-\varepsilon\kappa\xi\psi(\psi'+\psi)\big) + \big(4V(\psi)+\rho_m\big)\big(\psi'+6\xi\psi\big)\Big\} = 0. \end{aligned} \quad (10)$$

where the prime denotes differentiation with respect to the natural logarithm of the scale factor.

Now we can simply present equation (10) in the form of the autonomous dynamical system

$$\psi' = y, \quad (11a)$$

$$\begin{aligned} y' = &-y + \frac{\varepsilon y^2(1-6\xi)\big(y+6\xi\psi\big)}{\Big(\frac{6}{\kappa} - \varepsilon 6\xi(1-6\xi)\psi^2\Big)} - \frac{1}{2}\frac{\Big(\frac{6}{\kappa} - \varepsilon\big[(1-6\xi)y^2 + 6\xi(y+\psi)^2\big]\Big)}{\Big(\frac{6}{\kappa} - \varepsilon 6\xi(1-6\xi)\psi^2\Big)\Big(V(\psi)+\rho_r+\rho_m\Big)}\times \\ &\times\Big(\varepsilon\frac{6}{\kappa}V'(\psi)\big(1-\varepsilon\kappa\xi\psi(y+\psi)\big) + \big(4V(\psi)+\rho_m\big)\big(y+6\xi\psi\big)\Big). \end{aligned} \quad (11b)$$

TABLE I: Finite critical points and their characters.

	existence		
	$\varepsilon=+1$	$\varepsilon=-1$	eigenvalues
$\psi_0^2=\frac{1}{\varepsilon 6\xi}$, $y_0=0$	$\xi>0$	$\xi<0$	$\lambda_1=\frac{1}{2}(\varepsilon 3-5)\frac{1}{\psi_0}$, $\lambda_2=\frac{1}{2}(\varepsilon 3+5)\frac{1}{\psi_0}$
$\psi_0^2=-\frac{1}{\varepsilon 6\xi}$, $y_0=0$	$\xi<0$	$\xi>0$	$\lambda_{1,2}=-3(1-3\xi)\psi_0\pm\sqrt{-\varepsilon\frac{2}{\xi}(1-3\xi)(3-25\xi)}$
$\psi_0=0$, $y_0^2=\varepsilon$	$\forall\xi\in\mathbf{R}$	–	$\lambda_1=y_0$, $\lambda_2=2y_0$
$\psi_0^2=\frac{1}{\varepsilon 6\xi(1-6\xi)}$, $y_0=-\varepsilon\frac{1}{1-6\xi}\frac{1}{\psi_0}$	$0<\xi<\frac{1}{6}$	$\xi<0$ or $\xi>\frac{1}{6}$	$\lambda_1=-y_0$, $\lambda_2=-2y_0$

In what follows we will assume a quadratic potential function $V(\psi)=1/2m^2\psi^2$, and that there is no other form of matter than the scalar field, i.e that $\rho_r=\rho_m=0$. It is easy to notice that in this case the dynamics does not depend on the change of a sign of the potential $V(\psi)\to-V(\psi)$. Finally, the dynamical system is in the form

$$\alpha\frac{\mathrm{d}\psi}{\mathrm{d}\sigma}=y\psi\Big(\frac{6}{\kappa}-\varepsilon 6\xi(1-6\xi)\psi^2\Big), \tag{12a}$$

$$\alpha\frac{\mathrm{d}y}{\mathrm{d}\sigma}=-y\psi\Big(\frac{6}{\kappa}-\varepsilon 6\xi(1-6\xi)\psi^2\Big)+\varepsilon(1-6\xi)\psi y^2(y+6\xi\psi)- \\ -\Big(\frac{6}{\kappa}-\varepsilon\big[(1-6\xi)y^2+6\xi(y+\psi)^2\big]\Big)\Big(\varepsilon\frac{6}{\kappa}\big(1-\varepsilon\kappa\xi\psi(y+\psi)\big)+2\psi\big(y+6\xi\psi\big)\Big). \tag{12b}$$

where we have made the following "time" transformation

$$\alpha\frac{\mathrm{d}}{\mathrm{d}\sigma}=\psi\Big(\frac{6}{\kappa}-\varepsilon 6\xi(1-6\xi)\psi^2\Big)\frac{\mathrm{d}}{\mathrm{d}\ln(a)} \tag{13}$$

where the parameter α

$$\alpha=+1 \iff \psi\big(\frac{6}{\kappa}-\varepsilon 6\xi(1-6\xi)\psi^2\big)>0, \tag{14a}$$

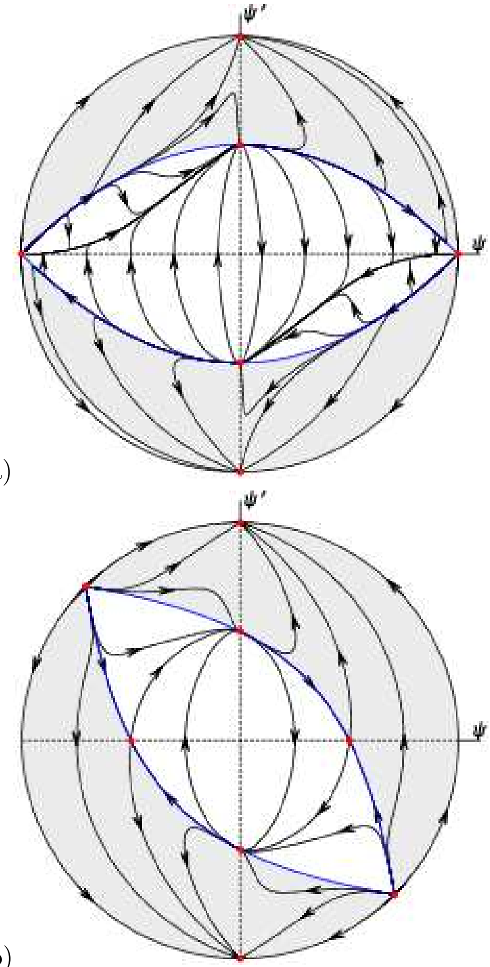
$$\alpha=-1 \iff \psi\big(\frac{6}{\kappa}-\varepsilon 6\xi(1-6\xi)\psi^2\big)<0. \tag{14b}$$

was introduced to preserve the orientation of the trajectories in this way that on all of the phase portraits direction of arrows indicate the direction of growth of the scale factor.

III. PHASE SPACE ANALYSIS OF DYNAMICS

In this section we present detailed discussion of the type of critical points of the dynamical system (12).

If we write the evolutional equations for the non-minimally coupled scalar field in the form of the dynamical system, the first step would be an identification of the critical points of the system. Physically they represent asymptotic (or stationary) states of the system under considerations and



a)

b)

FIG. 1: The phase portraits for the canonical scalar field $\varepsilon = 1$ for: a) minimal $\xi = 0$ and b) conformal $\xi = 1/6$ coupling. The shaded region denotes nonphysical part of the phase space for the strictly positive potential function. If the potential function is strictly negative the meaning of the regions is reversed. The shape of the border between the regions does not depend on the shape of the potential function. At the border of the physical region we have two symmetric critical points at the ψ axis for both cases. The value of H^2 at that points is finite. The presence of a saddle type critical point in the case b) is the effect of non-zero ξ.

mathematically correspond to vanishing r.h.s. of the system. The second step is a characterization of the type of critical point which can be performed after calculation of the eigenvalues of the linearization matrix calculated of this critical point. The critical points are usually represented by physically interesting solutions and these solutions can be attractors for trajectories in its neighborhood which evolve to it independently on the initial conditions. In the quintessence cosmology we are looking for the attractors, which give rise to solutions with desired property but we would like to know whether it is typical (generic) solution or exceptional (non-generic). This is a reason

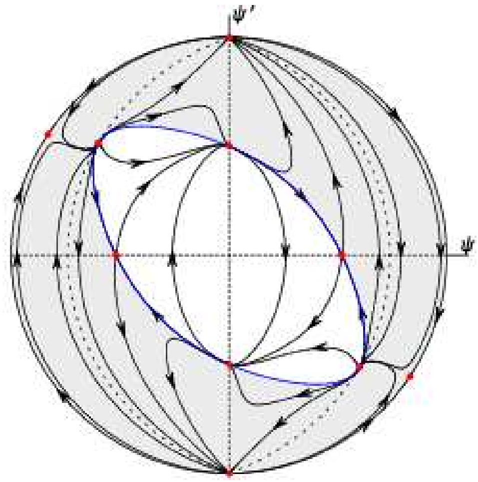
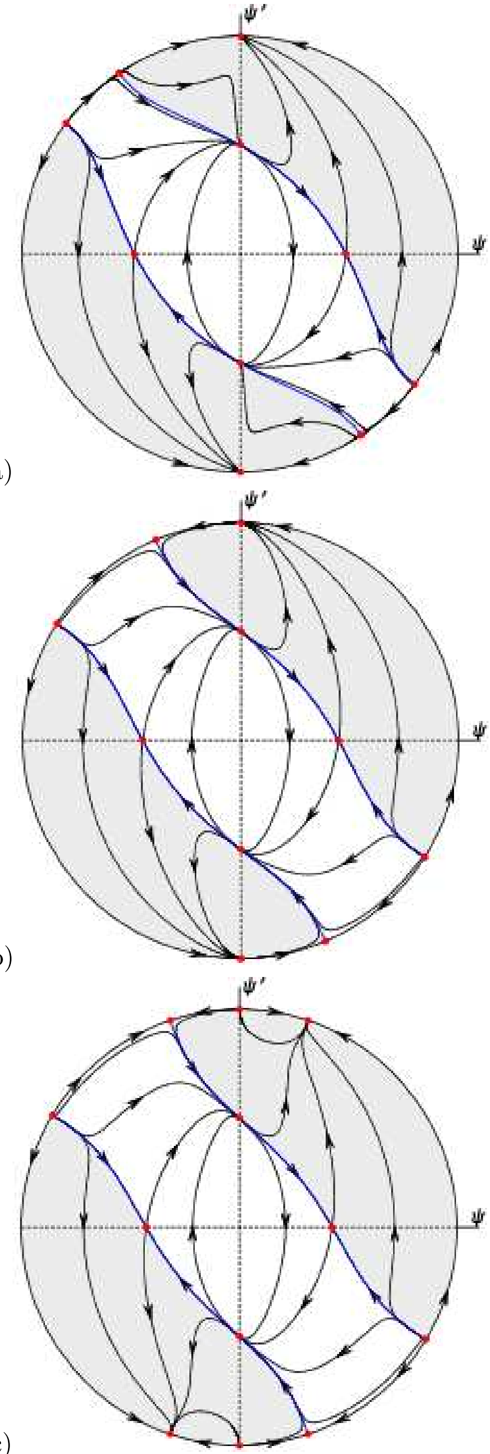


FIG. 2: The phase portrait for the canonical scalar field $\varepsilon = 1$ and coupling constant $0 < \xi < 1/6$. The shaded region is nonphysical: $H^2 < 0$ for $V(\psi) > 0$. There are three types of critical points at the finite domain of the phase space: 1) $\psi_0 = 0$, $y_0^2 = 1$ and $H^2 = \text{const.}$ which are of a stable or unstable node type; 2) $\psi_0^2 = 1/6\xi$, $y_0 = 0$, $H^2 = \infty$ of a saddle type 3) $\psi_0 \neq 0$, $y_0 \neq 0$, $H^2 = \infty$ of unstable node type for $V(\psi) > 0$ and stable node type for $V(\psi) < 0$ (shaded region). The dashed line denotes singularity of "time" transformation (10). In comparison with the phase portrait from Fig. 1 for conformal coupling we can see that both phase portraits are equivalent at the physical domain.

of our interest in the stability analysis of the critical points.

For full dynamical analysis investigation of the behavior of the trajectories at infinity is needed. It can be performed by construction of Poincaré sphere [30]. If we complete the phase plane by a circle at infinity which is a projection on the equator, then we receive the global phase portrait with a circle at infinity. In our case r.h.s. contain the non-minimal coupling constant ξ as a free parameter. The global phase portraits depend on the value of this parameter but for some ranges of the values of ξ phase portraits can be equivalent (indistinguishable from the dynamical system point of view). If we fix the value of non-minimal coupling, then one can study the influence of this parameter on the global dynamics. The equivalence of the phase portraits is established by homeomorphism preserving direction of time along the trajectories.

Critical point: $\psi_0 = 0$, $y_0^2 = \varepsilon$ exists only for the canonical scalar field $\varepsilon = +1$. Direct calculation of the Hubble function (8) at this point gives an undefined symbol $\frac{0}{0}$. It is why we use the linearized solutions in the vicinity of this critical point to show that the Hubble function at this point is finite



a)
b)
c)

FIG. 3: The phase portraits for the canonical scalar field $\varepsilon = 1$ and for the specific values of coupling constant: a) $\xi = 3/16$, b) $\xi = 1/4$, c) $\xi = 3/10$. In the cases a) and b) there are the critical points at infinity of a mixed type (multiple critical points). At the physical domain the phase portraits are topologically equivalent.

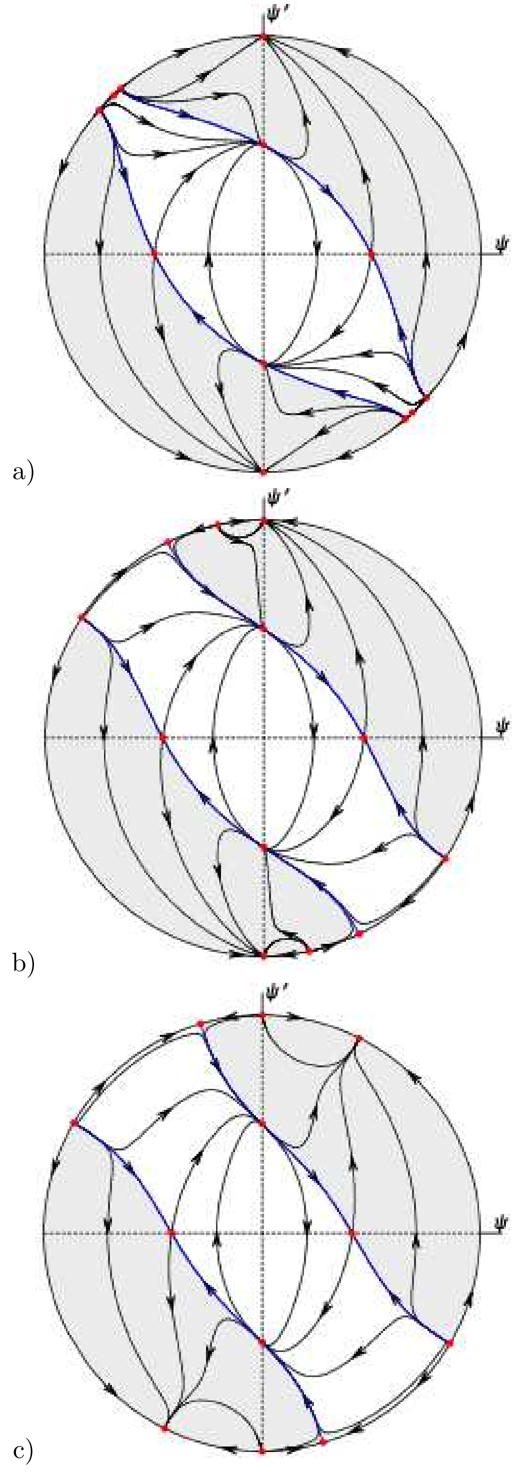


FIG. 4: The phase portraits for the canonical scalar field $\varepsilon = 1$ and values of coupling constant: a) $1/6 < \xi < 3/16$, b) $3/16 < \xi < 1/4$, c) $\xi = 1/3$.

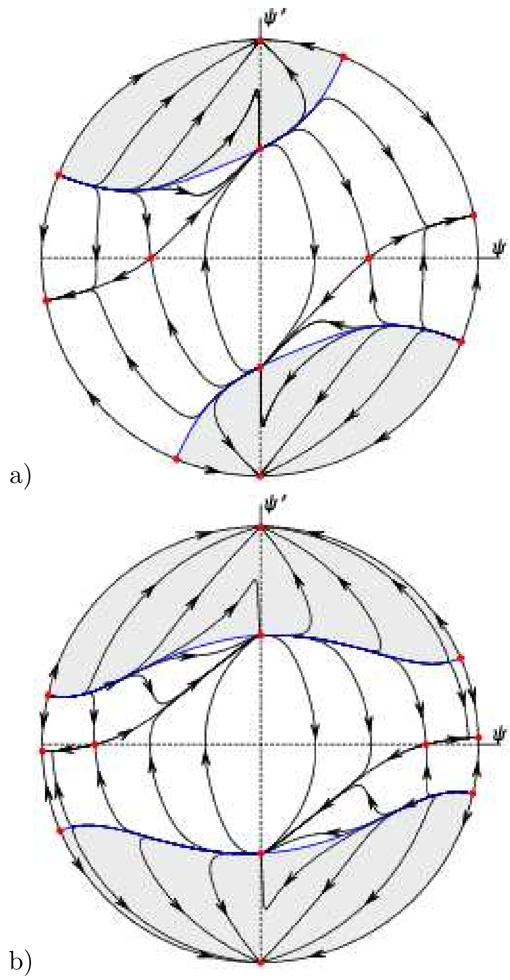


FIG. 5: The phase portraits for the canonical scalar field $\varepsilon = 1$ and negative coupling constant $\xi < 0$: in figure b) ξ is greater, but still negative, than in figure a). In the limit $\xi \to 0^-$ we receive phase portrait in Fig. 1a.

and depends on the initial conditions of the linearized solutions. They are in the form

$$\psi(\sigma) = \psi_{(\mathrm{i})} \exp{(\alpha\lambda_1\sigma)}, \tag{15a}$$

$$y(\sigma) = y_0 - 6\xi\psi_{(\mathrm{i})} \exp{(\alpha\lambda_1\sigma)} + \big(6\xi\psi_{(\mathrm{i})} + (y_{(\mathrm{i})} - y_0)\big) \exp{(\alpha\lambda_2\sigma)}, \tag{15b}$$

where $\lambda_1 = y_0$ and $\lambda_2 = 2y_0$ are eigenvalues of the linearization matrix calculated at this critical point, $\psi_{(\mathrm{i})}$ and $y_{(\mathrm{i})}$ are initial conditions and y_0 is a coordinate of the critical point.

Inserting those solution into the formula (8) we receive that the Hubble function in the vicinity of the critical point ($\psi_0 = 0$, $y_0^2 = \varepsilon$) is

$$H^2_{\mathrm{lin}} = m^2\psi^2_{(\mathrm{i})} \exp{(2\alpha\lambda_1\sigma)} \Big[- 6\xi(1-6\xi)\psi^2_{(\mathrm{i})} \exp{(2\alpha\lambda_1\sigma)} - 2y_0\big(6\xi\psi_{(\mathrm{i})} + (y_{(\mathrm{i})} - y_0)\big) \exp{(\alpha\lambda_2\sigma)} - \\ - \big(36\xi^2\psi^2_{(i)} + 12\xi\psi_{(i)}(y_{(i)} - y_0) + (y_{(i)} - y_0)^2\big) \exp{(2\alpha\lambda_2\sigma)}\Big]^{-1} \tag{16}$$

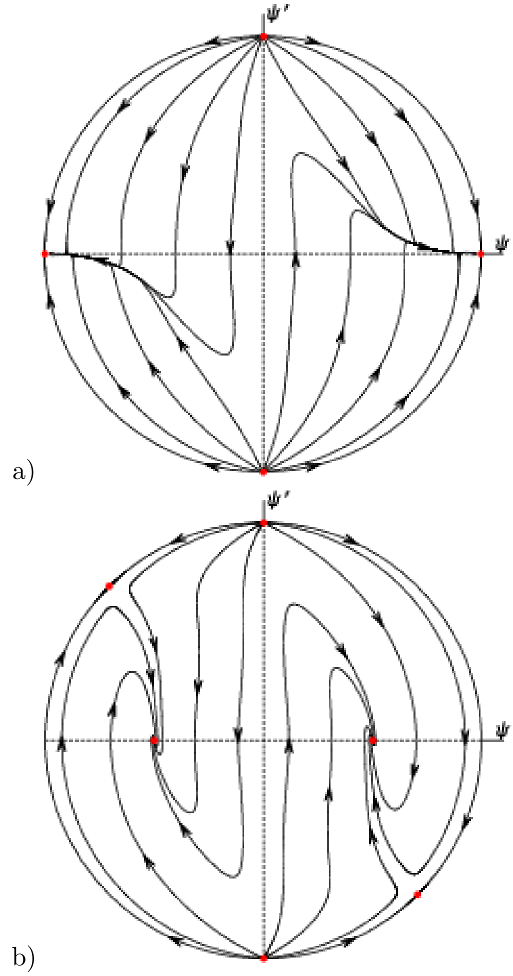
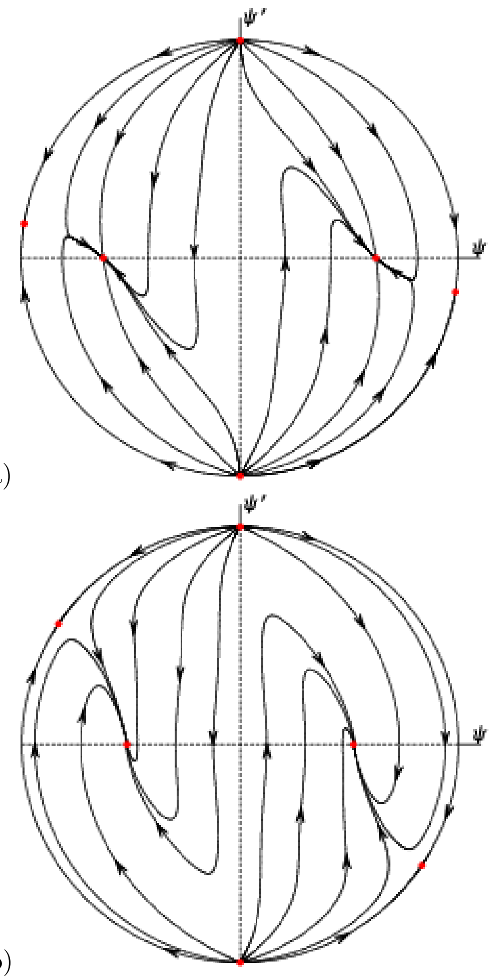


FIG. 6: The phase portraits for the phantom scalar field $\varepsilon = -1$ and: a) minimal $\xi = 0$ and b) conformal $\xi = 1/6$ coupling. All the phase space (ψ, ψ') is admissible only for the positive potential function. We can conclude, that for negative potential functions in the case of minimally or conformally coupled phantom scalar fields, the scale factor is not a monotonic function of the cosmological time. For the case b) a global attractor represents the de Sitter state with $w_{\psi} = -1$. There are two types of trajectories which tend to this attractor: 1) trajectories starting from $\psi = 0$, $\psi' = \pm\infty$ state, and 2) two single trajectories representing a separatrix of saddles at infinity (not shown).

then taking the limit value of this function for $\sigma \to \pm\infty$ (depending on the critical point $y_0 = \mp 1$) we receive

$$\lim H^2_{\text{lin}} = m^2 \frac{\psi^2_{(i)}}{-2y_0\big(6\xi\psi_{(i)} + (y_{(i)} - y_0)\big) - 6\xi(1-6\xi)\psi^2_{(i)}} \tag{17}$$

which is always a positive quantity. For the special values of minimal $\xi = 0$ (see Fig. 1a) and



a)

b)

FIG. 7: The global phase portraits for the phantom scalar field and values of coupling constant: a) $0 < \xi < 3/25$ and b) $3/25 < \xi < 1/6$. In the case a) in the finite domain the critical domain is of a stable node type and in the case b) of a focus type.

conformal coupling $\xi = 1/6$ (see Fig. 1b) the values of the Hubble function are

$$H^2_{\text{lin}} = m^2 \frac{\psi^2_{(i)}}{-2y_0(y_i - y_0)}, \qquad \text{for} \qquad \xi = 0,$$

$$H^2_{\text{lin}} = m^2 \frac{\psi^2_{(i)}}{-2y_0\big(\psi_{(i)} + (y_{(i)} - y_0)\big)}, \qquad \text{for} \qquad \xi = \frac{1}{6}.$$

Critical point: $\psi_0^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$, $y_0 = -\varepsilon \frac{1}{(1-6\xi)\psi_0}$ is very interesting because the Hubble function (8) at this point is singular $H^2 = \infty$ and $\dot{H} = (\frac{1}{2}H^2)' = \infty$.

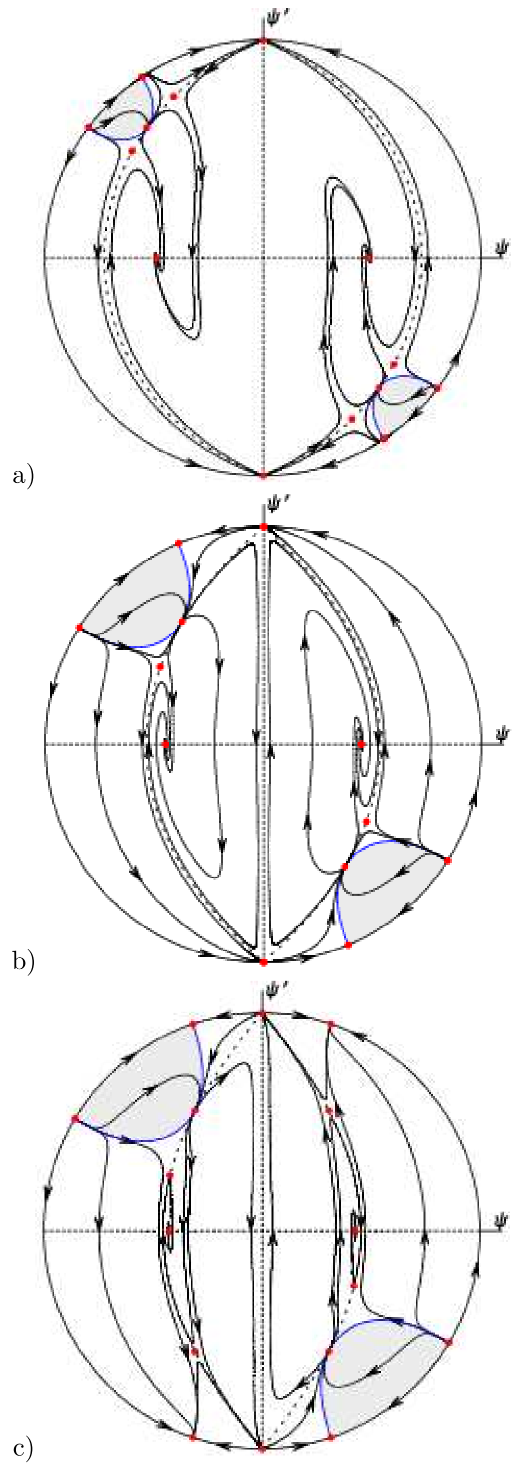


FIG. 8: The global phase portraits for the phantom scalar field $\varepsilon = -1$ and for the specific values of coupling constant: a) $\xi = 3/16$, b) $\xi = 1/4$, c) $\xi = 3/10$. In cases a) and b) one of the critical points at infinity is of a mixed type (multiple critical points) (see Fig. 3). On all figures one can see trajectories starting from the unstable node and landing at the stable focus as a generic scenario of route to the de Sitter state.

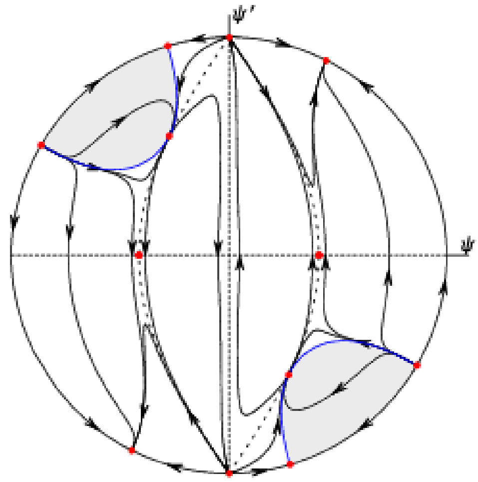


FIG. 9: The global phase portrait for the phantom scalar field $\varepsilon = -1$ and distinguished value of coupling constant $\xi = 1/3$. In this case critical point at the finite domain of the phase space is located at the line of singularities of the time transformation (13).

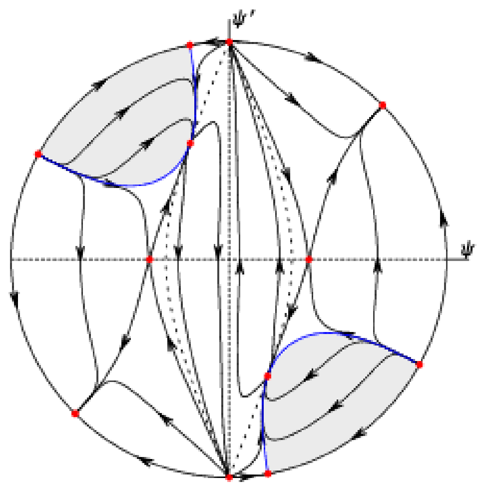


FIG. 10: The global phase portrait for the phantom scalar field $\varepsilon = -1$ and the values of coupling constant $\xi > 1/3$. The characteristic critical point of a focus type disappeared.

Linearized solutions in the vicinity of this critical points are

$$\psi(\sigma) = \psi_0 + \exp\left(\alpha\lambda_2\sigma\right)\left(\psi_{(i)} - \psi_0\right), \tag{18a}$$

$$y(\sigma) = y_0 + \exp\left(\alpha\lambda_1\sigma\right)\Big(2(1-3\xi)\left(\psi_{(i)} - \psi_0\right) + \left(y_{(i)} - y_0\right)\Big) - \exp\left(\alpha\lambda_2\sigma\right)\Big(2(1-3\xi)\left(\psi_{(i)} - \psi_0\right)\Big) \tag{18b}$$

where $\lambda_1 = -y_0$ and $\lambda_2 = -2y_0$ are eigenvalues of the linearization matrix at the critical point, $\psi_{(i)}$ and $y_{(i)}$ are initial conditions and ψ_0 and y_0 are coordinates of the critical point.

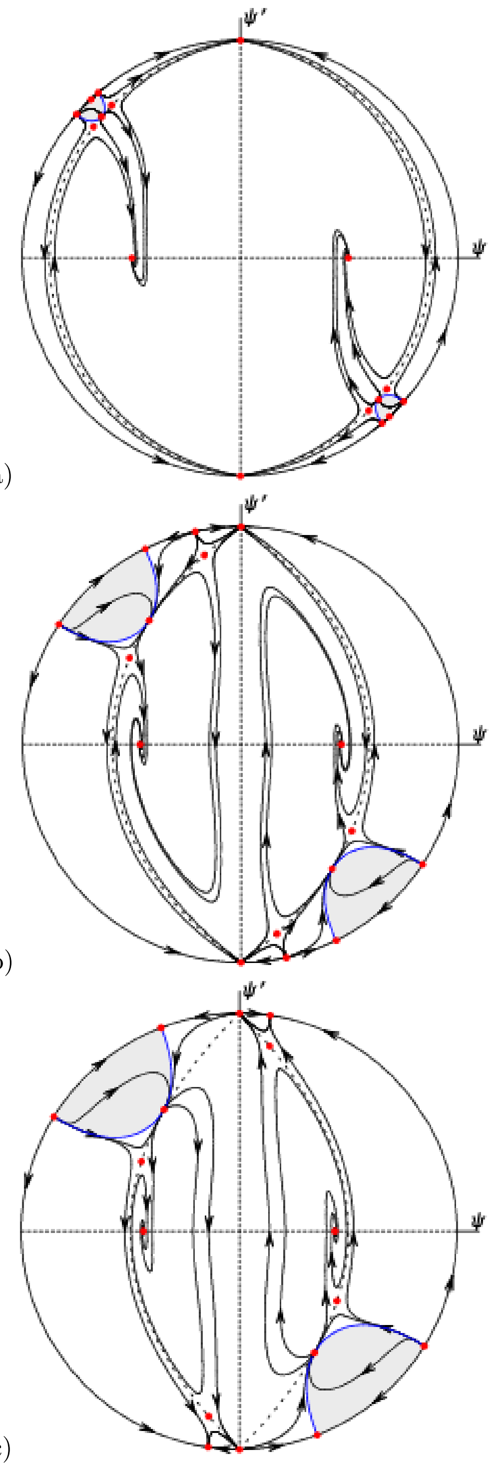


FIG. 11: The global phase portraits for the phantom scalar field $\varepsilon = -1$ and the values of coupling constant: a) $1/6 < \xi < 3/16$, b) $3/16 < \xi < 1/4$, c) $1/4 < \xi < 3/10$. In all cases there is present scenario of reaching the global attractor (a focus type critical point) from the unstable node. Note that in the case c) not all of the trajectories starting from an unstable node are reaching the de Sitter state, in contrast to cases a) and b).

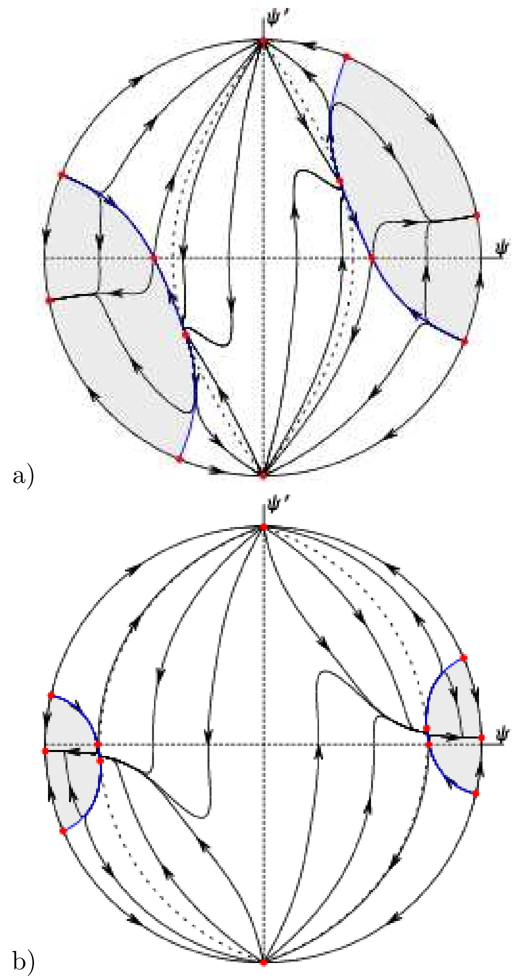


FIG. 12: The global phase portraits for the phantom scalar field $\varepsilon = -1$ and negative coupling constant $\xi < 0$: in figure b) ξ is greater, but still negative, than in figure a). It is easy to notice that in the limit $\xi \to 0^-$ we receive phase portrait from the Fig. 6a. for the phantom scalar field with minimal coupling.

Using time transformation (13) we can calculate the scale factor growth along the trajectory

$$\Delta \ln a = \int_0^\infty \psi(\sigma)\big(1 - \varepsilon 6\xi(1-6\xi)\psi(\sigma)^2\big)\mathrm{d}\sigma, \tag{19}$$

and the cosmological time growth

$$\Delta t = \int_{a_i}^{a_f} \frac{1}{H}\mathrm{d}\ln a = \int_0^\infty \frac{1}{H}\psi(\sigma)\big(1 - \varepsilon 6\xi(1-6\xi)\psi(\sigma)^2\big)\mathrm{d}\sigma. \tag{20}$$

Linearized solutions are good approximations of the original system in the vicinity of the critical point. In what follows we assume that $\big(\psi_{(i)} - \psi_0\big)^2 = \big(y_{(i)} - y_0\big)^2 = \big(\psi_{(i)} - \psi_0\big)\big(y_{(i)} - y_0\big) = 0$, $\alpha = 1$ and $y_0 > 0$ (see Fig. 12). Then

$$\Delta \ln a = \varepsilon(1-6\xi)\big(\psi_{(i)} - \psi_0\big)\psi_0 \tag{21}$$

and

$$\Delta t = -\frac{1}{\sqrt{m^2}}\varepsilon 12\xi(1-6\xi)\big(\psi_{(i)}-\psi_0\big)\psi_0\int_0^\infty \sqrt{A\exp{(-y_0\sigma)}+B\exp{(-2y_0\sigma)}}\exp{(-2y_0\sigma)}\mathrm{d}\sigma$$
$$= \frac{1}{\sqrt{m^2}}\varepsilon 12\xi(1-6\xi)\big(\psi_{(i)}-\psi_0\big)\frac{1}{y_0 24B^{5/2}}\Big\{\frac{3}{2}A^2\Big(\log A - 2\log{(\sqrt{B}+\sqrt{A+B})}\Big)-$$
$$- \sqrt{B(A+B)}(-A+2B)(3A+4B)\Big\} \tag{22}$$

where A and B are positive constants

$$A = -\varepsilon\big(y_0+12\xi\psi_0\big)\big(2(1-3\xi)(\psi_{(i)}-\psi_0)+(y_{(i)}-y_0)\big)$$

$$B = \varepsilon 4(1-6\xi)(y_0+3\xi\psi_0)(\psi_{(i)}-\psi_0)$$

For every case of existence of such a critical point (see Figs 2, 8, 9, 10, 11 for an unstable node and Fig. 12 for a stable node in the "physical region") growth of the scale factor and the cosmological time is finite.

Now we calculate the first derivative of the Hubble function (8) with respect to the cosmological time at this point

$$\dot{H} = \frac{1}{2}(H^2)' = \Big(\frac{V(\psi)}{1-\varepsilon[\psi'^2+12\xi\psi\psi'+6\xi\psi^2]}\Big)' \tag{23}$$

where a dot denotes differentiation with respect to the cosmological time and a prime with respect to the natural logarithm of the scale factor. Then after the elimination of second derivative of the field with respect to the natural logarithm of the scale factor we have

$$\dot{H} = -\frac{6\xi V'(\psi)\psi}{1-\varepsilon 6\xi(1-6\xi)\psi^2} - \frac{\varepsilon 2V(\psi)\big((1-6\xi)\psi'^2+(\psi'+6\xi\psi)^2\big)}{\big(1-\varepsilon 6\xi(1-6\xi)\psi^2\big)\big(1-\varepsilon[\psi'^2+12\xi\psi\psi'+6\xi\psi^2]\big)} \tag{24}$$

This expression at the critical point $\psi_0^2 = \frac{1}{\varepsilon 6\xi(1-6\xi)}$, $\psi_0' = -\varepsilon\frac{1}{1-6\xi}\frac{1}{\psi_0}$ is singular since the numerator is finite and the denominator is equal zero.

The trajectories starting form this critical point corresponding to the singularities of finite scale factor seems to be most interesting. For such state appearing for both, canonical and phantom scalar fields we have curvature singularity because Hubble parameter is infinite but the scale factor assumes finite value. They are typical because the critical point is an unstable node (see Fig. 2 for an unstable node for the canonical scalar field and and Figs. 8, 9, 10 and 11 for an unstable node and Fig. 12 for a stable node for the phantom scalar field).

IV. CONCLUSIONS

We study the dynamics of a scalar field with a simple quadratic potential function and non-minimally coupled to the gravity via $\xi R\psi^2$ term, where R is the Ricci scalar of the Robertson-Walker spacetime. We reduce dynamical problem to the autonomous dynamical system on the phase plane in the variables ψ and its derivative with respect to the natural logarithm of the scale factor. The constraint condition is solved in such a way that the final dynamical system is free from the constraint and is defined on the plane. We investigate the whole dynamics at the finite domain and at infinity. All the trajectories for all admissible initial conditions are classified, and critical points representing the asymptotic states (stationary solutions) are found. We explore generic evolutionary paths to find the stable de Sitter state as a global attractor and classify typical routes to this point. We study the effects of the canonical scalar field as well as the phantom scalar field. The following conclusions, as the results of our studies, can be drawn:

1. The cosmological models with the quadratic potential function and non-minimal coupling term $\xi R\psi^2$ are represented by a 2-dimensional autonomous dynamical system which is studied in details on the phase plane (ψ, ψ') (see Table I for the critical points at the finite domain). The shape of the physical region $H^2 \geq 0$ does not depend on the form of the potential function, but only on the value of the coupling constant ξ;

2. We investigated the fixed points of the dynamical system and their stability to find the generic evolutional scenarios. We have shown the existence of a finite scale factor singular point (both future and past) where the Hubble function as well as its first cosmological time derivative diverges $H^2 = \infty$ and $\dot{H} = \infty$;

3. For the phantom scalar field $\varepsilon = -1$ we found existence of a sink type critical point (i.e. a stable node or a focus, depending on the value of the parameter ξ) which represents the de Sitter solution. Only for $\xi < 1/6$ the evolutional paths avoid a past finite scale factor singularity (see for example Figs 6 and 8);

4. For the canonical scalar field $\varepsilon = 1$ we found that for $0 < \xi < 1/6$ exists the critical point which corresponds to a past finite scale factor singularity for models with $V(\psi) = \frac{1}{2}m^2\psi^2 > 0$. If $m^2 < 0$ then this point corresponds to a future finite scale factor singularity (see Fig. 2).

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