Investment delay and system dynamics

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We study the role of investment on dynamics of macroeconomic models. The delay of investment is understood as the time to build of capital goods. The result is delay differential equations applied to modelling the economic growth [3, 6]. The time lag in the investment is responsible for complexity of economic evolution in short and long run. The long run effects determine characteristics of economic development. The possibility of description of irregularities forced by delay in growth paths is considered. We also investigate optimal consumption in terms of advanced-retarded differential equations.

1. Introduction

We present two models of economic phenomena. First, the model of business cycle where a time delay in investment process causes endogenous cycles to reproduce economic fluctuations. Second, the model of economic growth where production lags are responsible for the irregularities on the long-run growth path. The idea of using delay differential equations to model business cycles was proposed by Kalecki [2]. He considered a gestation period between taking investment decisions and delivering capital goods, and showed formally that endogenous cycles can be generated. To study of nonlinear behaviour of models we use techniques of functional analysis and bifurcation theory. For determining the cyclic behaviour the Poincare-Andronov-Hopf bifurcation theorem is used which has been generalized for delay differential equations [1]. The Hopf theorem predicts bifurcation to a limit cycle when the real part of a pair of complex conjugate (with nonzero imaginary part) eigenvalues of the characteristic equation changes its sign from negative to positive as the bifurcation parameter is varied, and the derivative of the real part of the eigenvalues with respect to the bifurcation parameter is positive as real part passes through zero.

2. The model of business cycle

We formulate the Kalecki-Kaldor model of business cycle as the time-delay differential equation system [3, 5, 7, 8]

\[ \frac{dY}{dt} = \omega T(1 - \frac{Y}{k}) - kY, \]

\[ \frac{dk}{dt} = \alpha Y - \beta k - \delta k, \]

where \( Y \) is the investment and \( k \) is the saving function, \( \omega T \) is gross product, \( k \) is capital stock, \( \alpha \) is the adjustment coefficient in the goods market, \( \beta \) is the depreciation rate of the capital stock, and the time delay \( T \) is a constant average lag between investment decision and implementation of new capital.

The saving function \( k \) depends only on \( Y \) and is linear such that \( Sp = \gamma Y \in [0,1] \). Additionally, we assume that the investment function \( f(Y) \) is separable in respect to its two arguments and both are linear such that \( f_{1,2,3} \leq b_i \leq \beta \) if then \( f(Y, k) \) and \( k \) are compact such that \( Y \in [0, k] \). With these assumptions the Kalecki-Kaldor dynamical system has the form

\[ Y = \frac{\alpha Y - \beta k - \delta k}{\beta} + \delta k, \]

\[ k = \frac{\alpha Y - \beta k - \delta k}{\beta} \]

or equivalently

\[ Y(t) = f(Y(t) + k(t)) = 0, \]

where

\[ f(Y, k) = \frac{\alpha Y - \beta k - \delta k}{\beta}, \]

\[ g(Y, k) = \frac{\alpha Y - \beta k - \delta k}{\beta}. \]

If we assume a solution of \( f \) in the form \( Y(t) = \lambda Y \), we derive the eigenvalue equation

\[ \lambda^2 + A\lambda + B + D\lambda - M = 0, \]

where \( A, B \) and \( D \) are constant

\[ A = -\alpha - \beta - \delta, \]

\[ B = -\alpha + \beta, \]

\[ D = -\beta. \]

We assume the eigenvalue in the form \( \lambda = \sigma + \omega \) and write the real and imaginary parts of the eigenvalue equation as

\[ 2\sigma + \omega A - D\lambda - M = 0, \]

\[ 2\omega + \omega A - D\lambda - M = 0. \]

The limit cycle bifurcation occurs when the real part of a complex conjugate eigenvalues changes its sign from negative to positive. Putting \( \omega = 0 \) and \( \omega = \lambda \), we first solve eq. (9) then \( \alpha \) and \( \beta \)

\[ \frac{Y(t)}{\alpha Y - \beta k - \delta k} = \frac{\alpha Y - \beta k - \delta k}{\beta} = \frac{\delta k}{\beta} \]

where \( z = \frac{Y(t)}{\alpha Y - \beta k - \delta k} \) yields

\[ \frac{\partial}{\partial \lambda} \left( \frac{\gamma Y(t) - \beta k(t) - \delta k(t)}{\beta} \right) = \lambda - \left( \frac{\alpha Y(t) - \beta k(t) - \delta k(t)}{\beta} \right), \]

and it follows

\[ \frac{\partial}{\partial \lambda} \left( \frac{\gamma Y(t) - \beta k(t) - \delta k(t)}{\beta} \right) = \lambda - \left( \frac{\alpha Y(t) - \beta k(t) - \delta k(t)}{\beta} \right). \]

If we assume that \( \lambda D \geq 0 \) (always positive) then \( \frac{\partial}{\partial \lambda} \left( \frac{\gamma Y(t) - \beta k(t) - \delta k(t)}{\beta} \right) \) is the performed of the parameter \( \gamma Y(t) - \beta k(t) - \delta k(t) \) and \( \lambda(\gamma Y(t) - \beta k(t) - \delta k(t)) \) is the performed of the parameter \( \gamma Y(t) - \beta k(t) - \delta k(t) \).

Moreover we can determine the period of an orbit by

\[ P = \frac{2\pi}{\omega}, \]

In Fig. 1 it is shown the dependence of the bifurcation parameter \( t \) on \( T \). And the amplitude of this relation is equal \( 1/\omega \) and corresponds the value of \( \omega \). The right panel it is checked that the derivative of a real part of the eigenvalue with respect to \( T \) is positive for small \( y \).

3. The model of economic growth

Assuming that the rate of change of the capital stock at moment \( t \) is a function of the productive capital stock \( k(t - T) \), and capital stock depreciates at the rate \( \delta \in [0, 1] \) we obtain the following dynamic equation [9, 6]

\[ k = f(k(t - T)) = \delta k(t - T), \]

\[ k = f(k(t - T)) = \delta k(t - T), \]

where \( f(k) \) is a nonsequential production function that is continuous, increasing, and strictly concave in variable \( k \), and \( \alpha \in (0, 1) \) is a constant saving rate. Instead of an initial point value for an ordinary differential equation, the initial function \( f(k) \) is required which is defined over the range of time delayed by the delay.

Let start from a local stability analysis and consider the unique critical point \( x^* \) for

\[ s(x^*) = \frac{\partial f(x)}{\partial x} \]

Such a point exists if \( \frac{\partial f(x)}{\partial x} \) satisfies the Inada conditions. After the linearization of the right-hand side system (10) at the critical point we obtain

\[ \delta k(t - T) - \delta k(t - T) - \delta k(t - T) \]

To find the origin of the critical point by introducing the new variable \( e \), and then (11) can be rewritten as

\[ z = \frac{\partial f(x)}{\partial x} - \delta (k(t - T)), \]

To find the periodic cycle of the limit cycle is created by the Hopf bifurcation we first, find the bifurcation value of the time delay parameter, \( T \). For this purpose it is sufficient to show the existence of a unique pair of the complex conjugate solutions of (14) \( \lambda \).

4. Conclusion

We studied two economic models described by delay differential equations. The delay was the period of the starting and finishing the investment. Both in the model of business cycle and the model of economic growth the cycle behaviour can appear for some specific values of parameters due to the Hopf bifurcation where the time delay parameter is the bifurcation parameter.

References